Fast Energy Decay for Wave Equations with Variable Damping Coefficients in the 1-D Half Line

Ryo IKEHATA*and Takeshi KOMATSU

Department of Mathematics, Graduate School of Education, Hiroshima University

Higashi-Hiroshima 739-8524, Japan

June 17, 2015

Abstract

We derive fast decay estimates of the total energy for wave equations with localized variable damping coefficients, which are dealt with in the one dimensional half line $(0, \infty)$. The variable damping coefficient vanishes near the boundary $x = 0$, and is effective critically near spatial infinity $x = \infty$.

1 Introduction

We consider the 1-dimensional initial-boundary value problem for linear dissipative wave equation in the half line $(0, +\infty)$:

$$u_{tt}(t,x) - u_{xx}(t,x) + V(x)u_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times (0,\infty), \tag{1.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in (0,\infty), \tag{1.2}$$

$$u(t,0) = 0, \quad t \in (0,\infty), \tag{1.3}$$

where the initial data (u_0, u_1) are compactly supported with the size $R > 0$:

$$u_0 \in H_0^1(0,\infty), \quad u_1 \in L^2(0,\infty), \quad \text{supp } u_i \subset [0,R], (i = 0,1).$$

Let us first talk about several related results of the Cauchy problem in $\mathbf{R}^N$ of the equation:

$$u_{tt}(t,x) - \Delta u(t,x) + V(x)u_t(t,x) = 0. \tag{1.4}$$

In the case when $V(x) = \text{constant} > 0$, Matsumura [9] derived a historical L^p-L^q decay estimates of solutions of (1.4). Later on more detail investigations such as the asymptotic profiles of solutions were studied by Nishihara [14] and the references therein.

In the case of mixed problems of (1.4) in the exterior domain $\Omega \subset \mathbf{R}^N$ Nakao [13] and Ikehata [4] have derived precise decay estimates of the total energy for the initial data belonging to the energy class and weighted L^2-class, respectively. In these cases they have dealt with the localized damping coefficient $V(x)$, which is effective only near infinity such as $V(x) > V_0$ for $|x| \gg 1$, and the star-shaped boundary $\partial\Omega$. In particular, fast decay estimates of the total

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energy like $E(t) = O(t^{-2})$ was obtained in [4] by a special multiplier method developed in [6], which modified the Morawetz one [12]. Racke [15] also derived the L^p-L^q estimates for solutions to (1.4) with constant $V(x)$ in the exterior domain case by relying on so called the generalized Fourier transform (in fact, more general form of equations and systems are considered). For related results in the case when the damping is effective near infinity one can cite the papers written by Aloui-Ibrahim-Khenissi [1] and Daoulatli [2], where they studied the fast decay of the total and local energy under the so called GCC assumption due to Lebeau [8]. Exponential decay of the total energy for wave equations with mass (i.e., Klein-Gordon type) and localized damping terms was investigated by Zuazua [19] in the exterior domain case. Furthermore, by Watanabe [18] an effective application of the method developed in [4] is recently introduced, and there a global existence and decay estimates of solutions to a nonlinear Cauchy problem of the quasilinear wave equation with a localized damping near infinity are studied.

On the other hand, recently in [7] the critically decaying damping coefficient $V(x)$ near spatial infinity was considered for (1.4) in the whole space $\mathbf{R}^N$, in fact, they have studied so called the critical case with $\delta = 1$ below:

$$\frac{V_0}{(1 + |x|)^\delta} \leq V(x) \leq \frac{V_1}{(1 + |x|)^\delta} \quad x \in \mathbf{R}^N, \tag{1.5}$$

and they have announced the fact that $E(t) = O(t^{-N})$ if $V_0 > N$, while if $V_0 \leq N$, then (roughly speaking) $E(t) = O(t^{-V_0})$. In this connection, we call it sub-critical damping in the case of (1.5) with $\delta \in [0, 1)$, while in the case when $\delta > 1$, (1.5) is called as super-critical damping. The precise decay order of several norms of solutions for the Cauchy problem in $\mathbf{R}^N$ of (1.4) with (1.5) were investigated by Todorova-Yordanov [16] in the sub-critical case. Furthermore, in the super critical case it is well-known by Mochizuki [10] that the solution of the Cauchy problem of (1.4) is asymptotically free.

While, in the constant damping case with $V(x) = $ constant of (1.1)-(1.3) (i.e., problem in the half line) was studied in [3], and he derived $E(t) = O(t^{-2})$ $(t \to +\infty)$. In the localized damping case, it seems that there are no any previous research papers that are dealing with the half space problem and critical damping near infinity. In this connection, one notes that in [5] Ikehata-Inoue have derived $E(t) = O(t^{-1})$ as $t \to +\infty$ for $V_0 > 1$ (see also Mochizuki-Nakazawa [11] and Uesaka [17] for the topics on the energy decay property). In fact, they treated a more general nonlinear equation like

$$u_{tt}(t, x) - \Delta u(t, x) + V(x)u_t(t, x) + |u(t, x)|^{p-1}u(t, x) = 0,$$

in the critical damping case of (1.5).

So, a natural question arises whether one can derive fast decay result like $E(t) = O(t^{-2})$ in the case when $V(x)$ vanishes near the boundary and is effective critically (i.e., $\delta = 1$ in (1.5)) only near spatial infinity $x = \infty$. The main purpose of this paper is to answer this natural question by modifying a technical method originally developed in [5].

Our assumptions on $V(x)$ are as follows:

(**A**) $V \in L^\infty(0, \infty) \cap C([0, \infty))$, and there exist three constants

$$V_0 > \mathbf{2}, \quad V_1 \in [V_0, +\infty), \quad L_2 > 0$$

such that

$$\frac{V_0}{1 + x} \leq V(x) \leq \frac{V_1}{1 + x}, \quad x \in [L_2, +\infty),$$
$$0 \leq V(x) \quad (x \in [0, \infty)).$$

Our new results read as follows.

Theorem 1.1 *Let $V(x)$ satisfy the assumption* (**A**). *If the initial data $(u_0, u_1) \in H_0^1(0, \infty) \times L^2(0, \infty)$ further satisfies*

$$supp\, u_0 \cup supp\, u_1 \subset [0, R]$$

for some $R > 0$, then for the solution $u \in C([0, \infty); H_0^1(0, \infty)) \cap C^1([0, \infty); L^2(0, \infty))$ to problem $(1.1) - (1.3)$ *one has*

$$E(t) = O(t^{-2}), \quad (t \to \infty).$$

Remark 1.1 In the 1-D whole space case, if we consider the corresponding Cauchy problem, we can have

$$E(t) = O(t^{-1}), \quad (t \to \infty)$$

provided that $V_0 > 1$ (cf., [7], [17]). So, the results seem to reflect the half space property itself. This discovery of the new number $V_0 > \mathbf{2}$ in the half space case is one of our maximal contribution. Even if one takes $L_2 = 0$ formally, the obtained result in Theorem 1.1 is new.

Remark 1.2 In the case when $V_0 \leq 2$, we still do not know the exact rate of decay for $E(t)$. However, from [7, Theorem 4.1] one can state a conjecture that $E(t) = O(t^{-V_0})$ $(t \to +\infty)$ if $V_0 \leq 2$. Furthermore, the result will be generalized to the N-dimensional half space problems of (1.1)-(1.3). We can present a conjecture that if $V_0 > N + 1$, then $E(t) = O(t^{-(N+1)})$ $(t \to +\infty)$ in the half space case.

Remark 1.3 The assumption (**A**) implies that the damping coefficient $V(x)$ can vanish near the boundary $x = 0$, so the damping is effective only near infinity. But, the effectiveness of the damping $V(x)$ near infinity corresponds to so called the critical case of (1.5) with $\delta = 1$ (see [7]). In this sense, the result obtained in Theorem 1.1 is derived under two types of difficult situations, that is, one is vanishing damping, and the other is critical one.

This paper is organized as follows. In section 2 we shall prove Theorem 1.1 by relying on a multiplier method which was originally introduced by the first author's collaborative work [5] and [6].

Notation. Throughout this paper, $\| \cdot \|$ means the usual $L^2(\Omega)$-norm. The total energy $E(t)$ to the solution $u(t, x)$ of (1.1) is defined by

$$E(t) := \frac{1}{2}(\|u_t(t, \cdot)\|^2 + \|u_x(t, \cdot)\|^2).$$

Furthermore, one sets as a $L^2(0, \infty)$ inner product:

$$(u, v) := \int_0^{+\infty} u(x)v(x)dx.$$

A weighted function space is defined as follows: $u \in L^{1,1/2}(0, \infty)$ iff $u \in L^1(0, \infty)$ and

$$\|u\|_{1,1/2} := \int_0^\infty \sqrt{1+x}|u(x)|dx < +\infty.$$

Furthermore, one sets

$$X_1(0, \infty) := C([0, \infty); H_0^1(0, \infty)) \cap C^1([0, \infty); L^2(0, \infty)).$$

2 Proof of Results

We first prepare an important lemma, which is borrowed from [3, Lemma 2.1].

Lemma 2.1 *It holds that*

$$\sup_{0 \leq x < +\infty} \left(\frac{|u(x)|}{\sqrt{1+x}} \right) \leq \|u_x\|$$

for all $u \in H_0^1(0, \infty)$.

Lemma 2.2 *Let $u \in X_1(0, \infty)$ be the solution to problem $(1.1) - (1.3)$. Then it is true that*

$$\frac{d}{dt}\mathcal{E}(t) + \mathcal{F}(t) = 0, \quad (t \geq 0)$$

where

$$\mathcal{E}(t) = \frac{1}{2}\frac{d}{dt}\int_0^\infty [f(u_t^2 + u_x^2) + 2guu_t + (gV - g_t)u^2 + 2hu_xu_t]dx,$$

$$\mathcal{F}(t) = \frac{1}{2}\int_0^\infty (2fV - f_t - 2g + h_x)u_t^2 dx + \frac{1}{2}\int_0^\infty (2g - f_t + h_x)u_x^2 dx$$

$$+ \frac{1}{2}\int_0^\infty (g_{tt} - g_tV)u^2 dx + \int_0^\infty (hV - h_t)u_xu_t dx - \frac{1}{2}\int_0^\infty \frac{d}{dx}(hu_x)dx.$$

Moreover, $f = f(t)$, $g = g(t)$ and $h = h(t, x)$ are all smooth functions specified later on.

Proof. Since one considers the weak solutions $u(t, x)$ of the problem (1.1)-(1.3), we can assume to check the desired identity that the solution $u(t, x)$ is sufficiently smooth, and vanishes for large $x \gg 1$.

Step 1. Multiplying the both sides of (1.1) by fu_t and rearranging it one can get:

$$fu_tu_{tt} - fu_tu_{xx} + fVu_t^2$$

$$= f\frac{1}{2}\frac{d}{dt}u_t^2 - \frac{d}{dx}(fu_tu_x) + \frac{1}{2}\frac{d}{dt}fu_x^2 - \frac{1}{2}f_tu_x^2 + fVu_t^2$$

$$= \frac{1}{2}\frac{d}{dt}[f(u_t^2 + u_x^2)] + (fV - \frac{1}{2}f_t)u_t^2 - \frac{1}{2}f_tu_x^2 - \frac{d}{dx}(fu_tu_x) = 0.$$

Step 2. Multiplying the both sides of (1.1) by gu and rearranging it one has:

$$guu_{tt} - guu_{xx} + gVu_tu$$

$$= g\frac{d}{dt}uu_t - gu_t^2 - \frac{d}{dx}(guu_x) + gu_x^2 + gV\frac{1}{2}\frac{d}{dt}u^2$$

$$= \frac{d}{dt}guu_t - g_tuu_t - gu_t^2 - \frac{d}{dx}(guu_x) + gu_x^2 + \frac{1}{2}\frac{d}{dt}(gVu^2) - \frac{1}{2}g_tVu^2$$

$$= \frac{d}{dt}(guu_t) - \frac{1}{2}\frac{d}{dt}(g_tu^2) + \frac{1}{2}g_{tt}u^2 - gu_t^2 - \frac{d}{dx}(guu_x) + gu_x^2 + \frac{1}{2}\frac{d}{dt}(gVu^2) - \frac{1}{2}g_tVu^2 = 0.$$

Step 3. Multiplying the both sides of (1.1) by hu_x and rearranging it one obtains:

$$hu_xu_{tt} - hu_xu_{xx} + hu_xVu_t$$

$$= h(\frac{d}{dt}u_xu_t) - h\frac{d}{dt}(u_x)u_t - \frac{1}{2}h\frac{d}{dx}(u_x^2) + hVu_tu_x$$

$$= \frac{d}{dt}(hu_xu_t) - h_tu_xu_t - \frac{1}{2}h\frac{d}{dx}(u_t^2) - \frac{1}{2}\frac{d}{dx}(hu_x^2) + \frac{1}{2}h_xu_x^2 + hVu_tu_x$$

$$= \frac{d}{dt}(hu_t u_x) - h_t u_x u_t - \frac{1}{2}\frac{d}{dx}(hu_t^2) + \frac{1}{2}h_x u_t^2 - \frac{1}{2}\frac{d}{dx}(hu_x^2) + \frac{1}{2}h_x u_x^2 + hVu_t u_x = 0.$$

Step 4. Add all final identities from Step 1 to Step 3 and integrate over $[0, \infty)$. Then, it follows from the boundary condition (1.3) that one can get:

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty [f(u_t^2 + u_x^2) + 2guu_t + (gV - g_t)u^2 + 2hu_t u_x]dx$$

$$+\frac{1}{2}\int_0^\infty (2fV - f_t - 2g + h_x)u_t^2 dx + \frac{1}{2}\int_0^\infty (2g - f_t + h_x)u_x^2 dx$$

$$+\frac{1}{2}\int_0^\infty (g_{tt} - g_t V)u^2 dx + \int_0^\infty (hV - h_t)u_x u_t dx - \frac{1}{2}\int_0^\infty \frac{d}{dx}(hu_x^2)dx = 0,$$

which implies the desired identity. $\square$

Since the finite speed of propagation property can be applied to the solution $u(t, x)$ of the corresponding problem $(1.1) - (1.3)$, in order to estimate the functions $\mathcal{E}(t)$ and $\mathcal{F}(t)$ it suffices to consider all the spatial integrand over the closed interval $[0, R + t]$ $(t \geq 0)$.

Now, because of the assumption $(\mathbf{A})$ such as $V(L_2) > 0$ and the continuity of $V(x)$, there exist constants $L_1 \in [0, L_2)$ and $V_m > 0$ such that

$$V(x) \geq V_m \quad (x \in [L_1, L_2]).$$

By using these constants we choose the functions $f(t)$, $g(t)$ and $h(t, x)$ as follow:

$$f(t) = \epsilon_1(1 + t)^2, \quad g(t) = \epsilon_2(1 + t), \quad h(t, x) = \epsilon_3(1 + t)\phi(x),$$

where the **monotone increasing** function $\phi \in C^\infty([0, \infty))$ can be defined to satisfy

$$\phi(x) = \begin{cases} (1 + x) & (0 \leq x < L_1), \\ 1 + L_2 & (L_2 \leq x), \end{cases} \tag{2.1}$$

where $\epsilon_i > 0$ $(i = 1, 2, 3)$ are some constants determined later on.

Lemma 2.3 *Let f, g and h be defined by (2.1). Then one has the following estimates: for each $t \geq 0$,*
(i) $\int_0^\infty (hV - h_t)u_t u_x dx \geq -\frac{k}{2}\int_0^\infty hV u_t^2 dx - \frac{1}{2k}\int_0^\infty hV u_x^2 dx - \frac{1}{2}\int_0^\infty h_t u_t^2 dx - \frac{1}{2}\int_0^\infty h_t u_x^2 dx,$
(ii) $\int_0^\infty \frac{d}{dx}(hu_x^2)dx \leq 0,$
where $k > 0$ is a constant.

Proof. (i): This is easily derived from the following inequality with some positive number p:

$$|u_t u_x| \leq \frac{1}{2p}u_t^2 + \frac{p}{2}u_x^2.$$

(ii): Because of the finite speed of propagation property, one has

$$\int_0^\infty \frac{d}{dx}(hu_x^2)dx = h(t, \infty)u_x(t, \infty)^2 - h(t, 0)u_x(t, 0)^2$$

$$= -h(t, 0)u_x(t, 0)^2 \leq 0. \quad \square$$

Thus, it follows from Lemmas 2.2 and 2.3 that for $t \geq 0$

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty [f(u_t^2 + u_x^2) + 2guu_t + (gV - g_t)u^2 + 2hu_t u_x]dx$$

$$+\frac{1}{2}\int_0^\infty (2fV - f_t - 2g + h_x - khV - h_t)u_t^2\,dx$$

$$+\frac{1}{2}\int_0^\infty (2g - f_t + h_x - \frac{1}{k}hV - h_t)u_x^2\,dx + \frac{1}{2}\int_0^\infty (g_{tt} - g_t V)u^2\,dx \le 0. \qquad (2.2)$$

Furthermore, one can get the estimates below:

Lemma 2.4 *Let f, g and h be defined by (2.1), and assume* (**A**). *If all parameters ϵ_i $(i = 1, 2, 3)$ are well-chosen, then there exists a large $t_0 > 0$ such that for all $t \ge t_0 \gg 0$, one has*
(iii) $2f(t)V(x) - f_t(t) - 2g(t) + h_x(t, x) - kh(t, x)V(x) - h_t(t, x) > 0, \quad x \in [0, \infty),$
(iv) $2g(t) - f_t(t) + h_x(t, x) - \dfrac{1}{k}h(t, x)V(x) - h_t(t, x) > 0, \quad x \in [0, \infty)$
with some constant $k > 0$.

Proof. We first check (iii) by separating the integrand of x into 3 parts $[0, L_1]$, $[L_1, L_2]$, and $[L_2, +\infty)$.
(iii) The case $\underline{0 \le x \le L_1}$:

$$2fV - f_t - 2g + h_x - khV - h_t$$

$$= 2\epsilon_1(1+t)^2 V(x) - 2\epsilon_1(1+t) - 2\epsilon_2(1+t) + \epsilon_3(1+t) - k\epsilon_3(1+t)(1+x)V(x) - \epsilon_3(1+x)$$

$$\ge (1+t)^2\{2\epsilon_1 - k\epsilon_3\frac{1+L_1}{1+t}\}V(x) + (1+t)\{\epsilon_3 - 2\epsilon_1 - 2\epsilon_2 - \epsilon_3\frac{1+L_1}{1+t}\}.$$

The case $\underline{L_1 \le x \le L_2}$:

$$2fV - f_t - 2g + h_x - khV - h_t$$

$$= 2\epsilon_1(1+t)^2 V(x) - 2\epsilon_1(1+t) - 2\epsilon_2(1+t) + \epsilon_3(1+t)\phi'(x) - k\epsilon_3(1+t)\phi(x)V(x) - \epsilon_3\phi(x)$$

$$\ge (1+t)^2\{2\epsilon_1 V_m - 2\epsilon_1\frac{1}{1+t} - 2\epsilon_2\frac{1}{1+t} - k\epsilon_3 V_M\frac{1+L_2}{1+t} - \epsilon_3\frac{1+L_2}{(1+t)^2}\},$$

where $V_M := \sup\{V(x) | 0 \le x \le L_2\} > 0$, and we have just used the fact that $\phi'(x) \ge 0$.

The case $\underline{L_2 \le x}$: Here we use the finite speed of propagation property of the solution below:

$$2fV - f_t - 2g + h_x - khV - lh_t$$

$$= 2\epsilon_1(1+t)^2 V(x) - 2\epsilon_1(1+t) - 2\epsilon_2(1+t) - k\epsilon_3(1+t)(1+L_2)V(x) - \epsilon_3(1+L_2)$$

$$\ge 2\epsilon_1(1+t)^2\frac{V_0}{1+x} - 2\epsilon_1(1+t) - 2\epsilon_2(1+t) - k\epsilon_3(1+t)(1+L_2)\frac{V_1}{1+x} - \epsilon_3(1+L_2)$$

$$\ge \frac{(1+t)^2}{1+x}\{2\epsilon_1 V_0 - 2\epsilon_1\frac{1+R+t}{1+t} - 2\epsilon_2\frac{1+R+t}{1+t} - k\epsilon_3(1+L_2)\frac{V_1}{1+t} - \epsilon_3(1+L_2)\frac{1+R+t}{(1+t)^2}\}.$$

Note that in this region, $\phi'(x) = 0$.

Next one can check the condition (iv) similarly to the case (iii).
(iv) The case $\underline{0 \le x \le L_1}$:

$$2g - f_t + h_x - \frac{1}{k}hV - h_t$$

$$= 2\epsilon_2(1+t) - 2\epsilon_1(1+t) + \epsilon_3(1+t) - \frac{\epsilon_3}{k}(1+t)(1+x)V(x) - \epsilon_3(1+x)$$

$$\geq (1+t)\{2\epsilon_2 - 2\epsilon_1 + \epsilon_3 - \frac{\epsilon_3}{k}(1+L_1)V_M - \epsilon_3\frac{1+L_1}{1+t}\}.$$

The case $\underline{L_1 \leq x \leq L_2}$:

$$2g - f_t + h_x - \frac{1}{k}hV - h_t$$

$$= 2\epsilon_2(1+t) - 2\epsilon_1(1+t) + \epsilon_3(1+t)\phi'(x) - \frac{\epsilon_3}{k}(1+t)\phi(x)V(x) - \epsilon_3\phi(x)$$

$$\geq (1+t)\{2\epsilon_2 - 2\epsilon_1 - \frac{\epsilon_3}{k}(1+L_2)V_M - \epsilon_3\frac{1+L_2}{1+t}\}.$$

The case $\underline{L_2 \leq x}$:

$$2g - f_t + h_x - \frac{1}{k}hV - h_t$$

$$= 2\epsilon_2(1+t) - 2\epsilon_1(1+t) - \frac{\epsilon_3}{k}(1+t)(1+L_2)V(x) - \frac{\epsilon_3}{l}(1+L_2)$$

$$\geq (1+t)\{2\epsilon_2 - 2\epsilon_1 - \frac{\epsilon_2}{k}(1+L_2)\frac{V_1}{1+x} - \epsilon_2\frac{1+L_2}{1+t}\}$$

$$\geq (1+t)\{2\epsilon_2 - 2\epsilon_1 - \frac{\epsilon_3}{k}V_1 - \epsilon_2\frac{1+L_2}{1+t}\}.$$

So, in order to obtain (iii) and (iv) for large $t \gg 1$ the following five conditions should be satisfied to ϵ_i $(i = 1, 2, 3)$ and $k > 0$:

$$\epsilon_3 - 2\epsilon_1 - 2\epsilon_2 > 0, \tag{2.3}$$

$$2\epsilon_1 V_0 - 2\epsilon_1 - 2\epsilon_2 > 0, \tag{2.4}$$

$$2\epsilon_2 - 2\epsilon_1 + \epsilon_3 - \frac{\epsilon_3}{k}(1+L_1)V_M > 0, \tag{2.5}$$

$$2\epsilon_2 - 2\epsilon_1 - \frac{\epsilon_3}{k}(1+L_2)V_M > 0, \tag{2.6}$$

$$2\epsilon_2 - 2\epsilon_1 - \frac{\epsilon_3}{k}V_1 > 0. \tag{2.7}$$

(2.3) and (2.4) come from the check of (iii), while (2.5)-(2.7) have its origin in the case when we check (iv) as $t \to +\infty$.

We need to look for the constants ϵ_1, ϵ_2, ϵ_3, $k > 0$ satisfying these five conditions (2.3)-(2.7) above.

First, the condition (2.5) and (2.6) and (2.7) can be unified by the following one condition (2.8):

$$2\epsilon_2 - 2\epsilon_1 + \epsilon_3 - \frac{\epsilon_3}{k}(1+L_1)V_M \geq 2\epsilon_2 - 2\epsilon_1 - \frac{\epsilon_3}{k}(1+L_2)V_M$$

$$\geq 2\epsilon_2 - 2\epsilon_1 - \frac{\epsilon_3}{k}V^* > 0, \tag{2.8}$$

where

$$V^* := \max\{(1+L_2)V_M, V_1\} > 0.$$

Thus it suffices to check only 3 conditions (2.3), (2.4) and (2.8) above. However, for its purpose it is enough to choose all constants $\epsilon_i > 0$ $(i = 1, 2, 3)$ and $k > 0$ as follows:

$$\epsilon_1 := 1, \quad \epsilon_2 := \frac{V_0}{2}, \quad \epsilon_3 := 2V_0,$$

and

$$k := \frac{4V_0 V^*}{V_0 - 2},$$

where the assumption $V_0 > 2$ is essentially used. Therefore, one has the desired estimates if one takes $t \gg 1$ sufficiently large. $\quad\square$

Next lemma is a direct consequence of (2.2) and Lemma 2.4.

Lemma 2.5 *Let $u \in X_1(0, \infty)$ be the solution to problem $(1.1) - (1.3)$, and f, g, h be defined by (2.1). Under the condition $(\mathbf{A})$, the following estimate holds true:*

$$\frac{d}{dt}\{f(t)E(t) + g(t)(u_t, u) + 2(hu_x, u_t)\} \leq \frac{1}{2}\frac{d}{dt}\int_0^\infty (g_t - gV)u^2 dx + \frac{1}{2}\int_0^\infty (g_t V - g_{tt})u^2 dx \quad (2.9)$$

for all $t \geq t_0$, where $t_0 \gg 1$ is a fixed time defined in Lemma 2.4.

Then we integrate the both sides of (2.9) over $[t_0, t]$:

$$f(t)E(t) + g(t)(u_t, u) + 2(hu_x, u_t)$$

$$\leq f(t_0)E(t_0) + g(t_0)(u_t(t_0), u_t(t_0)) + 2(h(t_0)u_x(t_0), u_t(t_0))$$

$$+ \frac{1}{2}\int_0^\infty (g_t - gV)u^2 dx - \frac{1}{2}\int_0^\infty (g_t(t_0) - g(t_0)V)u^2(t_0)dx + \frac{1}{2}\int_{t_0}^t \int_0^\infty (g_t V - g_{tt})u^2 dx ds$$

$$= C + \frac{1}{2}\int_0^\infty (g_t - gV)u^2 dx + \frac{1}{2}\int_{t_0}^t \int_0^\infty (g_t V - g_{tt})u^2 dx ds, \quad\quad (2.10)$$

where the constant $C > 0$, which is independent from $t \geq t_0$, is defined by

$$C := f(t_0)E(t_0) + g(t_0)(u_t(t_0), u_t(t_0)) + 2(h(t_0)u_x(t_0), u_t(t_0))$$

$$- \frac{1}{2}\int_0^\infty (g_t(t_0) - g(t_0)V)u^2(t_0)dx.$$

Furthermore, we have the following estimates.

Lemma 2.6 *Let g be defined by (2.1). The the smooth function g satisfies the following two estimates:*

$$(v) \quad g_t - gV(x) \leq C_1, \quad x \in [0, \infty), \quad t \geq 0,$$

$$(vi) \quad g_t V - g_{tt} \leq C_2 V(x), \quad x \in [0, \infty), \quad t \geq 0,$$

where $C_i > 0$ $(i = 1, 2)$ are some constants.

Proof. The proof can be easily checked, so we omit it. $\quad\square$

On the other hand, we shall prepare the following crucial lemma based on the one dimensional Hardy-Sobolev inequality in the half space case, which is stated in Lemma 2.1.

Lemma 2.7 *Let $u \in X_1(0, \infty)$ be the solution to problem $(1.1) - (1.3)$. Then it is true that*

$$\|u(t, \cdot)\|^2 + \int_0^t \int_0^\infty V(x)|u(s, x)|^2 dx ds \leq C(\|u_0\|^2 + \|(V(\cdot)u_0 + u_1)\|_{1,1/2}^2), \tag{2.11}$$

provided that $\|(V(\cdot)u_0 + u_1)\|_{1,1/2} < +\infty$.

Proof. The original idea comes from [6]. We introduce an auxiliary function

$$w(t, x) := \int_0^t u(s, x) ds.$$

Then $w(t, x)$ satisfies

$$w_{tt} - w_{xx} + V(x)w_t = V(x)u_0 + u_1, \quad (t, x) \in (0, \infty) \times (0, \infty), \tag{2.12}$$

$$w(0, x) = 0, \quad w_t(0, x) = u_0(x), \quad x \in (0, \infty). \tag{2.13}$$

Multiplying (2.12) by w_t and integrating over $[0, t] \times [0, \infty)$ we get

$$\frac{1}{2}(\|w_t(t, \cdot)\|^2 + \|w_x(t, \cdot)\|^2) + \int_0^t \|\sqrt{V(\cdot)}w_s(s, \cdot)\|^2 ds$$

$$= \frac{1}{2}\|u_0\|^2 + \int_0^t (V(\cdot)u_0 + u_1, w_s) ds. \tag{2.14}$$

Next step is to use Lemma 2.1 to obtain a series of inequalities below:

$$\int_0^t (V(\cdot)u_0 + u_1, w_s) ds = \int_0^t \frac{d}{ds}(V(\cdot)u_0 + u_1, w) ds$$

$$\leq \int_0^\infty \sqrt{1+x}|V(x)u_0 + u_1| \frac{|w(t, x)|}{\sqrt{1+x}} dx$$

$$\leq \left(\sup_{x \in [0, \infty)} \frac{|w(t, x)|}{\sqrt{1+x}} \right) \|V(\cdot)u_0 + u_1\|_{1,1/2}$$

$$\leq \frac{1}{4}\|w_x\|^2 + \|V(\cdot)u_0 + u_1\|_{1,1/2}^2. \tag{2.15}$$

Combining (2.14) with (2.15) we can derive

$$\frac{1}{2}\|w_t(t, \cdot)\|^2 + \frac{1}{4}\|w_x(t, \cdot)\|^2 + \int_0^t \int_0^\infty V(x)w_t(s, x) dx ds$$

$$\leq \frac{1}{2}\|u_0\|^2 + \|V(\cdot)u_0 + u_1\|_{1,1/2}^2.$$

The desired estimate follows from the estimate above and the fact that $w_t = u$. $\quad \square$

It follows from (2.10) and Lemmas 2.6 and 2.7 that there exists a constant $C > 0$ such that

$$f(t)E(t) + g(t)(u(t, \cdot), u_t(t, \cdot)) + 2(hu_x, u_t) \leq C \quad (t \geq t_0), \tag{2.16}$$

provided that $\|(V(\cdot)u_0 + u_1)\|_{1,1/2} < +\infty$.

Finally, we can derive the following lemma.

Lemma 2.8 *Let h be defined by (2.1). Then, for all $t \geq t_0 \gg 1$ it is true that*

$$f(t)E(t) + 2(hu_x, u_t) \geq Cf(t)E(t),$$

where $C > 0$ is a constant.

Proof. Indeed, one has

$$f(t)E(t) + 2(hu_x, u_t) \geq \frac{1}{2}\int_0^\infty f(t)(u_t^2 + u_x^2)dx - \int_0^\infty h(t, x)(u_x^2 + u_t^2)dx$$

$$\geq \frac{1}{2}\int_0^\infty (f(t) - 2h(t, x))(u_x^2 + u_t^2)dx.$$

On the other hand, if necessarily, by choosing $t_0 \gg 1$ further large enough, one can derive the following estimates for $t \geq t_0$:

$$f(t) - 2h(t, x) \geq \epsilon_1(1 + t)^2 - 2\epsilon_3(1 + t)(1 + L_2)$$

$$= (1 + t)^2\{\epsilon_1 - \frac{2\epsilon_3(1 + L_2)}{1 + t}\} \geq C(1 + t)^2 \geq Cf(t),$$

with some constant $C > 0$. Here we have just used the monotonicity of the function $\phi(x)$ closely related with the definition of the function $h(t, x)$. $\quad\square$

Now we can finalize the proof of Theorem 1.1.

Proof of Theorem 1.1. We first note that one can use Lemma 2.7 because one can check $\|(V(\cdot)u_0 + u_1)\|_{1,1/2} < +\infty$ under the assumption on the initial data stated in Theorem 1.1. Thus, by using the Schwarz inequality, (2.16) and Lemma 2.8 we get

$$C(1 + t)^2E(t) \leq g(t)\|u(t, \cdot)\|\|u_t(t, \cdot)\| + C \leq Cg(t)\sqrt{E(t)} + C, \quad t \geq t_0.$$

Furthermore, if we set $X(t) = \sqrt{E(t)}$ for $t \in [0, +\infty)$, then one has

$$f(t)X(t)^2 - Cg(t)X(t) - C \leq 0, \quad t \geq t_0. \tag{2.17}$$

By solving the quadratic inequality (2.17) for $X(t)$ we have

$$\sqrt{E(t)} \leq \frac{Cg(t) + \sqrt{C^2g(t)^2 + 4Cf(t)}}{2f(t)} \quad (t \geq t_0).$$

This inequality leads to

$$E(t) \leq C\left(\frac{g(t)}{f(t)}\right)^2 + C\left(\frac{1}{f(t)}\right), \quad t \geq t_0,$$

which implies the desired decay estimates. $\quad\square$

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